Kuhn-Tucker

$$\text{Minimize}_{(x_1,x_2)} \quad (x_1-4)^2+(x_2-4)^2$$

subject to:

$$2x_1+3x_2 \geq 6$$
$$-3x_1-2x_2 \geq -12.$$

Solution

Note that the minimum (not restricted) of the objective function is at the point $(x_1, x_2) = (4, 4)$, so the restricted minimum must be located "as close as possible" to this point. The Kuhn-Tucker Lagrangian for this constrained optimization problem is

$$L(x_1, x_2, \lambda_1, \lambda_2) = (x_1 - 4)^2 + (x_2 - 4)^2 + \lambda_1(6 - 2x_1 - 3x_2) - \lambda_2(-12 + 3x_1 + 2x_2).$$

The Kuhn-Tucker conditions are

$$\frac{\partial L}{\partial x_1} = 2(x_1 - 4) - 2\lambda_1 - 3\lambda_2 \geq 0,$$

$$x_1\frac{\partial L}{\partial x_1} = x_1[2(x_1 - 4) - 2\lambda_1 - 3\lambda_2] = 0,$$

$$\frac{\partial L}{\partial x_2} = 2(x_2 - 4) - 3\lambda_1 - 2\lambda_2 \geq 0,$$

$$x_2\frac{\partial L}{\partial x_2} = x_2[2(x_2 - 4) - 3\lambda_1 - 2\lambda_2] = 0,$$

$$\frac{\partial L}{\partial \lambda_1} = 6 - 2x_1 - 3x_2 \leq 0,$$

$$\lambda_1\frac{\partial L}{\partial \lambda_1} = \lambda_1(6 - 2x_1 - 3x_2) = 0,$$

$$\frac{\partial L}{\partial \lambda_2} = 12 - 3x_1 - 2x_2 \leq 0,$$

$$\lambda_2\frac{\partial L}{\partial \lambda_2} = \lambda_2(12 - 3x_1 - 2x_2) = 0.$$

Together with the non-negativity conditions $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$.

If $x_1 = 0$ and $x_2 \neq 0$

$$2(0 - 4) - 2\lambda_1 - 3\lambda_2 \geq 0$$

$$-8 - 2\lambda_1 - 3\lambda_2 \geq 0$$

Then

$$2(x_2 - 4) - 3\lambda_1 - 2\lambda_2 = 0$$
$$2x_2 - 8 - 3\lambda_1 - 2\lambda_2 = 0$$

- If both constraints are inactive then $\lambda_1 = 0$ and $\lambda_2 = 0$:

  $$-8 \geq 0$$

  Which is contradictory

- If both constraints are active then $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$:

  $$6 - 3x_2 = 0$$
  $$2 = x_2$$

  And

  $$12 - 2x_2 = 0$$
  $$6 = x_2$$

  Which is contradictory

- If the first constraint is active and the second is inactive, then $\lambda_1 \geq 0$ (with which $\frac{\partial L}{\partial \lambda_1} = 0$) and $\lambda_2 = 0$ (and $\frac{\partial L}{\partial \lambda_2} \leq 0$). In this case, the condition $\dfrac{\partial L}{\partial \lambda_1} = 0$ is expressed as

$$6 - 2x_1 - 3x_2 = 0$$
$$x_2 = 2$$

But this implies:

$$\frac{\partial L}{\partial \lambda_2} = 12 - 3 * 0 - 2 * 2 \leq 0$$

$$\frac{\partial L}{\partial \lambda_2} = 8 \leq 0$$

Which is contradictory

- If the first constraint is inactive and the second active, it follows that $\lambda_1 = 0$ (and $\frac{\partial L}{\partial \lambda_1} \leq 0$) $\lambda_2 \geq 0$ (and $\frac{\partial L}{\partial \lambda_2} = 0$). In this case, the condition $\frac{\partial L}{\partial \lambda_2} = 0$ become

$$12 - 3x_1 - 2x_2 = 0$$
$$12 - 3 * 0 - 2x_2 = 0$$
$$x_2 = 6$$

Then

$$\frac{\partial L}{\partial x_2} = 2(6 - 4) - 3 * 0 - 2\lambda_2 = 0$$
$$4 = 2\lambda_2$$
$$\lambda_2 = 2$$

Then

$$\frac{\partial L}{\partial x_1} = 2(0 - 4) - 2 * 0 - 3 * 2$$
$$\frac{\partial L}{\partial x_1} = -8 - 6 = -14$$

Which contradicts the condition that $\frac{\partial L}{\partial x_1} \geq 0$

If $x_1 \neq 0$ and $x_2 = 0$

- If both constraints are inactive then $\lambda_1 = 0$ and $\lambda_2 = 0$: Then

$$\frac{\partial L}{\partial x_2} = 2(0 - 4) - 3 * 0 - 2 * 0 = -8$$

Which contradicts the condition that $\frac{\partial L}{\partial x_2} \geq 0$

- If both constraints are active then $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$:

$$6 - 2x_1 - 0 = 0$$

Then

$$x_1 = 3$$

But

$$\frac{\partial L}{\partial \lambda_2} = 12 - 3 * 3 - 2 * 0 = 9$$

But this contradicts $\frac{\partial L}{\partial \lambda_2} \leq 0$

- If the first constraint is active and the second is inactive, then $\lambda_1 \geq 0$ (with which $\frac{\partial L}{\partial \lambda_1} = 0$) and $\lambda_2 = 0$ (and $\frac{\partial L}{\partial \lambda_2} \leq 0$). In this case, the condition $\dfrac{\partial L}{\partial \lambda_1} = 0$ is expressed as

$$6 - 2x_1 - 3 * 0 = 0$$
$$x_1 = 3$$

Then

$$\frac{\partial L}{\partial \lambda_2} = 12 - 3 * 3 - 2 * 0 = 3$$

But this contradicts $\frac{\partial L}{\partial \lambda_2} \leq 0$

- If the first constraint is inactive and the second active, it follows that $\lambda_1 = 0$ (and $\frac{\partial L}{\partial \lambda_1} \leq 0$) $\lambda_2 \geq 0$ (and $\frac{\partial L}{\partial \lambda_2} = 0$). In this case, the condition $\frac{\partial L}{\partial \lambda_2} = 0$ become

$$12 - 3x_1 - 2x_2 = 0$$
$$12 - 3 * x_1 - 2 * 0 = 0$$
$$x_1 = 4$$

Then

$$\frac{\partial L}{\partial x_1} = 2(4 - 4) - 2 * 0 - 3\lambda_2 = 0$$

$$\lambda_2 = 0$$

But this implies

$$\frac{\partial L}{\partial x_2} = 2(x_2 - 4) - 3\lambda_1 - 2\lambda_2 < 0$$

Which contradicts

$$\frac{\partial L}{\partial x_2} = 2(x_2 - 4) - 3\lambda_1 - 2\lambda_2 \geq 0$$

If $x_1 = 0$ and $x_2 = 0$

Then the first restriction isn't satisfied

If $x_1 \neq 0$ and $x_2 \neq 0$

The complementary slackness conditions:

$$x_1 \frac{\partial L}{\partial x_1} = x_1[2(x_1 - 4) - 2\lambda_1 - 3\lambda_2] = 0,$$

$$x_2 \frac{\partial L}{\partial x_2} = x_2[2(x_2 - 4) - 3\lambda_1 - 2\lambda_2] = 0,$$

imply that $\frac{\partial L}{\partial x_1} = 0$ and $\frac{\partial L}{\partial x_2} = 0$. From here on we evaluate all the cases:

- If both constraints are inactive, $\lambda_1 = 0$ (and $\frac{\partial L}{\partial \lambda_1} \leq 0$) and $\lambda_2 = 0$ (and $\frac{\partial L}{\partial \lambda_2} \leq 0$), the conditions $\frac{\partial L}{\partial x_1} = 0$ and $\frac{\partial L}{\partial x_2} = 0$ are rewritten, respectively, as

$$2(x_1 - 4) = 0$$

and

$$2(x_2 - 4) = 0.$$

Solving this system results in the point (4, 4) with $\lambda_1 = 0$ and $\lambda_2 = 0$ as a candidate critical point for a minimum. But this does not satisfy the following conditions

$$\lambda_1 \frac{\partial L}{\partial \lambda_1} = \lambda_1(6 - 2x_1 - 3x_2) = 0,$$

$$\lambda_2 \frac{\partial L}{\partial \lambda_2} = \lambda_2(12 - 3x_1 - 2x_2) = 0.$$

- If the first constraint is active and the second is inactive, then $\lambda_1 \geq 0$ (with which $\frac{\partial L}{\partial \lambda_1} = 0$) and $\lambda_2 = 0$ (and $\frac{\partial L}{\partial \lambda_2} \leq 0$). In this case, the conditions $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$ and $\frac{\partial L}{\partial \lambda_1} = 0$ are expressed as

$$2(x_1 - 4) - 2\lambda_1 = 0,$$

$$2(x_2 - 4) - 3\lambda_1 = 0$$

and

$$6 - 2x_1 = 0,$$

respectively. Solving, the point $(x_1, x_2) = (\frac{7}{2}, 17)$ is obtained with $\lambda_1 = 3$. However, this point does not satisfy the inactive constraint.

- If the first constraint is inactive and the second active, it follows that $\lambda_1 = 0$ (and $\frac{\partial L}{\partial \lambda_1} \leq 0$) $\lambda_2 \geq 0$ (and $\frac{\partial L}{\partial \lambda_2} = 0$). In this case, the conditions $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$ and $\frac{\partial L}{\partial \lambda_2} = 0$ become

$$2(x_1 - 4) - 3\lambda_2 = 0,$$

$$2(x_2 - 4) - 2\lambda_2 = 0$$

and

$$12 - 3x_1 - 2x_2 = 0,$$

respectively. **From this, the result is $(x_1, x_2) = \left(\frac{28}{13}, \frac{36}{13}\right)$ with $\lambda_2 = \frac{16}{13}$ as a candidate for a minimum.**

- If both constraints are active, then $\lambda_1 \geq 0$ (and $\frac{\partial L}{\partial \lambda_1} = 0$) and $\lambda_2 \geq 0$ (and $\frac{\partial L}{\partial \lambda_2} = 0$). In this case, $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$, $\frac{\partial L}{\partial \lambda_1} = 0$ and $\frac{\partial L}{\partial \lambda_2} = 0$ derive in

$$2(x_1 - 4) - 2\lambda_1 - 3\lambda_2 = 0,$$

$$2(x_2 - 4) - 3\lambda_1 - 2\lambda_2 = 0,$$

$$6 - 2x_1 - 3x_2 = 0$$

and

$$12 - 3x_1 - 2x_2 = 0.$$

Solving the above system, we get $(x_1, x_2) = \left(\frac{24}{5}, -\frac{6}{5}\right)$ with $\lambda_1 = -\frac{172}{25}$ and $\lambda_2 = \frac{128}{25}$. However, this critical point is not a candidate for an optimum because it does not satisfy the condition $\lambda_1 \geq 0$.

The solution is therefore the point $(x_1, x_2) = \left(\frac{28}{13}, \frac{36}{13}\right)$ together with $(\lambda_1, \lambda_2) = \left(0, \frac{16}{13}\right)$ in which the objective function adopts the value $\frac{832}{169}$.